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Ray propagation in antisotropic inhomogeneous media

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Abstract. We derive the eikonal and transport equations, in the transverse magnetic (TM) and electric (TE) representation, for an electric anisotropic and inhomegeneous media by using the limit of geometrical optics. We also calculate the Poynting vector and analyse the consequences of the Poynting theorem for our system.

We study the Lagrangian representation for interpreting the trajectories and show that the TM ray trajectory in a uniaxial liquid crystal is that of the 2D projection of a 3D propagating ray in an isotropic medium, constrained by a surface. As an application of the formalism, we explicitly exhibit these surfaces for the radial and bipolar configuration in nematic droplets.

1. Introduction

The general treatment for ray tracing in inhomogeneous but isotropic media, based on the eikonal equation [1], provides an excellent description and insight for understanding light propagation. This phenomenon in media which are in addition anisotropic is a difficult problem which has been extensively studied in recent years for planar geometries. In fact, for the case of a plane wave and layered media, different procedures have been used for solving this problem. One of the most widely utilized methods in uniaxial systems is the Barreman's [2] 4×4 matrix formalism which was shown to be equivalent to Maxwell's equation for linear propagation. Some other approaches based on the well known geometrical optics approximation, have also been applied to describe beam propagation in planar geometries [3, 4]. Additionally, much work has been done in studying anisotropic systems by using more restricted approaches like the quasi-isotropic approximation of geometrical optics. An excellent review on this topic is given in [5]. There are also some more general procedures for describing both acoustic [6] and optics [7] of anisotropic media by using tensor eikonal, however, they do not take advantage of the complete representation given by transverse electric (TE) and transverse magnetic (TM) modes, which considerably simplify the treatment of nonmagnetic media.

The purpose of this work is to study the ray propagation in a general linear nonmagnetic media by using the TM and TE mode representation. More specifically, we shall derive the TM and TE eikonal equations and analyse their ray trajectories in their corresponding Lagrangian representation.

To this end, this paper is organized as follows. In section 2 we deduce from Maxwell's equations the TE and TM eikonal equations, In section 3 we derive the corresponding Lagrangians of the TE and TM rays, and state some analogies with isotropic media. In section 4 we calculate the energy density and Poynting vector and generalize the intensity law for anisotropic media. In section 5 we obtain the transport equations and derive an expression

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for the field amplitudes in terms of the optical path. In section 6 we illustrate these analogies with two examples in nematic droplets and we summarize our work.

2. Eikonal equations

We consider a general time-harmonic field in a nonconducting inhomogeneous and anisotropic medium. In regions free from currents and charges, Maxwell's equations are given by

$$\nabla \times \vec{H} + \mathrm{i}k_0 \overleftrightarrow{\epsilon} \cdot \vec{E} = 0 \qquad \nabla \cdot (\mu \vec{H}) = 0 \tag{1}$$

$$\nabla \times \vec{E} - \mathrm{i}k_0 \mu \vec{H} = 0 \qquad \nabla \cdot (\overleftrightarrow{\epsilon} \cdot \vec{E}) = 0 \tag{2}$$

where $k_0 = \omega/c$, μ is the magnetic susceptibility and $\overleftrightarrow{\epsilon}$ is the dielectric tensor. Following the usual procedure we assume the following trial form for the fields

$$\vec{E} = \vec{e}(\vec{r})\mathrm{e}^{\mathrm{i}k_0 l W(\vec{r})} \qquad \vec{H} = \vec{h}(\vec{r})\mathrm{e}^{\mathrm{i}k_0 l W(\vec{r})} \tag{3}$$

where $lW(\vec{r})$ is the characteristic function of Hamilton, which is equal to the difference in optical paths of a ray propagating between two fixed point in the medium, and $\vec{e}(\vec{r})$, $\vec{h}(\vec{r})$ are vector functions of the position. Substituting equation (3) into equations (1), (2) and taking the limit of the geometrical optics , $k_0 l \gg 1$, we have

$$\nabla W \times \vec{h} + \overleftrightarrow{\epsilon} \cdot \vec{e} = 0 \tag{4}$$

$$\nabla W \times \vec{e} - \mu \vec{h} = 0 \tag{5}$$

$$\nabla W \cdot \overleftrightarrow{\epsilon} \cdot \vec{e} = 0 \tag{6}$$

and

$$\nabla W \cdot \vec{h} = 0. \tag{7}$$

These equations show that the anisotropic nature of the medium expressed by the tensor $\overleftrightarrow{\epsilon}$, makes W sensible for the polarization of the fields $\vec{e}$ and $\vec{h}$; in a similar way as happens for anisotropic elastic media [8]. Here, we shall use an alternative approach, that is exclusive for electromagnetic system and consists of taking advantage of the complete representation given by the sets of TE and TM modes [9] for which the only electric or magnetic component, respectively, of these sets is transverse. We first consider the TE modes whose only magnetic component is $\vec{h}$. Inserting $\vec{h}$ from equation (5) into (4), yields

$$(\nabla W_{\mathrm{TE}} \times \hat{e})^2 = \mu \epsilon_{ee} \tag{8}$$

where $\hat{e} = \vec{e}/e$ and $\epsilon_{ee} = \hat{e} \cdot \overleftrightarrow{\epsilon} \cdot \hat{e}$. Then, for the TM modes, we solve equation (4) for $\vec{e}$ and insert it in equation (5), to obtain

$$(\nabla W_{\mathrm{TM}} \times \hat{h}) \cdot \overleftrightarrow{\epsilon}^{-1} \cdot (\nabla W_{\mathrm{TM}} \times \hat{h}) = \mu \tag{9}$$

where $\hat{h} = \vec{h}/h$ and the superscript -1 indicates the inverse tensor. Equations (8) and (9) are the so-called eikonal or Hamilton–Jacobi equations corresponding to the TE and TM modes. We can simplify both these equations by expressing them in terms of an orthogonal coordinate system $\{q_1, q_2, q_3\}$. If q_1 is the coordinate along $\hat{h}$ and $\hat{e}$, equations (8) and (9) can be rewritten as

$$\frac{1}{h_2^2}\left(\frac{\partial W_{\mathrm{TE}}}{\partial q_2}\right)^2 + \frac{1}{h_3^2}\left(\frac{\partial W_{\mathrm{TE}}}{\partial q_3}\right)^2 = \mu \epsilon_{11} \tag{10}$$

and

$$\frac{\epsilon_{22}^{-1}}{h_3^2}\left(\frac{\partial W_{\mathrm{TM}}}{\partial q_3}\right)^2 - 2\frac{\epsilon_{23}^{-1}}{h_2 h_3}\left(\frac{\partial W_{\mathrm{TM}}}{\partial q_2}\right)\left(\frac{\partial W_{\mathrm{TM}}}{\partial q_3}\right) + \frac{\epsilon_{33}^{-1}}{h_2^2}\left(\frac{\partial W_{\mathrm{TM}}}{\partial q_2}\right)^2 = \mu \qquad (11)$$

where h_i, $(i = 2, 3)$ are the scale factors and ϵ_{ij}^{-1} are the elements of $\overleftrightarrow{\epsilon}^{-1}$. Notice that, on the one hand, equation (10) shows that the TE rays propagate as in an isotropic medium with an index refraction $n^2 = \mu\epsilon_{11}$; which is a consequence of having a unique electric component. On the other hand, equation (11) has a different structure which implies a distinct behaviour for the TM rays. Similar and more complicated eikonal equations have been found in the context of geometrical acoustics where there exists three independent polarizations, one quasi-longitudinal and two quasi-transverse, that in the general case, cannot be decoupled [10]. Next we analyse the Lagrangian representations of the eikonal equations.

3. Lagrange dynamics

The Hamiltonian representation of equations (10) and (11) is obtained by rewriting them in terms of the variables $p_i = \frac{\partial W}{\partial q_i}$, $(i = 1, 2, 3)$, which are known as the ray components. It leads to

$$\frac{p_2^2}{h_2^2} + \frac{p_3^2}{h_3^2} = \mu\epsilon_{11} \qquad (12)$$

and

$$\frac{\epsilon_{33}^{-1}}{h_2^2}p_2^2 - 2\frac{\epsilon_{23}^{-1}}{h_2 h_3}p_2 p_3 + \frac{\epsilon_{22}^{-1}}{h_3^2}p_3^2 = \mu. \qquad (13)$$

It should be mentioned that from the Hamiltonian $\mathcal{H} = \mu$, we can calculate the four first-order diferential equations known as Hamilton's equation of which three are independent, since equation (13) allows us to write one in terms of the others. From these equations and their boundary conditions we can determine the ray trajectories for a particular system as was performed in previous works [11, 12]. Here, we shall study the Lagrangian formulation of the TM modes which provide us with a useful insight into the understanding of the behaviour of its ray trajectories. Hamilton's equation yields

$$(\dot{q}_2\dot{q}_3) = 2\begin{bmatrix} \frac{\epsilon_{33}^{-1}}{h_2^2} & -\frac{\epsilon_{23}^{-1}}{h_2h_3} \\ -\frac{\epsilon_{23}^{-1}}{h_2h_3} & \frac{\epsilon_{22}^{-1}}{h_3^2} \end{bmatrix}\begin{pmatrix} p_2 \\ p_3 \end{pmatrix}.$$

Hence, the Lagrangian $\mathcal{L}$ associated to $\mathcal{H}$ is obtained by using the Legendre transform. It leads to

$$\mathcal{L} = \epsilon_{33}^{-1}h_2^2\dot{q}_2^2 - 2\epsilon_{23}^{-1}h_2h_3\dot{q}_2\dot{q}_3 + \epsilon_{22}^{-1}h_3^2\dot{q}_3^2 \qquad (14)$$

where $\dot{q}_i = \mathrm{d}q_i/\mathrm{d}\tau$ are the components of a tangent vector to the trajectory. A ray whose Lagrangian is given by equation (14) does not experience a position dependent refraction index, instead it has an anisotropic kinetic energy in a curved space. However, for the case of a uniaxial liquid crystal [14] where $\epsilon_{ij} = \epsilon_\perp\delta_{ij} + \epsilon_a n_i n_j$, we can interpret the system as the 2D projection in a plane described by q_2 and q_3, of a 3D ray propagation whose degrees of freedom are q_2, q_3 and z and which is confined by a constriction of the form $\mathrm{d}z = f_2\mathrm{d}q_2 + f_3\mathrm{d}q_3$, where f_2 and f_3 are two functions of q_2 and q_3. This allows us to write the Lagrangian of the system in the following way:

$$\epsilon_\perp\mathcal{L} = \dot{z}^2 + h_2^2\dot{q}_2^2 + h_3^2\dot{q}_3^2 - \lambda(\dot{z} - f_2(q_1, q_3)\dot{q}_2 - f_3(q_2, q_3)\dot{q}_3) \qquad (15)$$

where λ is the Lagrange multiplier. Here, f_2 and f_3 are to be determined by substituting z in terms of q_2 and q_3 given by the constriction and by comparing its terms with the ones of equation (14). This leads to the explicit expression for the constriction given by

$$dz = -\sqrt{\frac{\epsilon_a}{\epsilon_\parallel}}(n_3 h_2 dq_2 - n_2 h_3 dq_3). \tag{16}$$

By applying the Lagrange equation to the variables λ and z, we find that λ is given by

$$\lambda = \lambda_0 - 2\sqrt{\frac{\epsilon_a}{\epsilon_\perp}}\vec{v}_\perp \cdot \hat{n}_\perp \tag{17}$$

where λ_0 is an integration constant and $\vec{v}_\perp = h_2\dot{q}_2\hat{e}_2 + h_3\dot{q}_3\hat{e}_3$ is the velocity component in the $q_1 - q_2$ plane, and $\hat{n}_\perp = n_1\hat{e}_1 - n_2\hat{e}_2$.

Because the constriction given by equation (16) depends on the configuration angle ψ, in general terms dz is not an exact differential $\oint dz \neq 0$, and as a consequence, it is not always possible to find a surface of the form $z = z(q_1, q_2)$, which means that equation (16) is a nonholomic constriction [13]. However, from Pfaffian theory [15], we know that for a differential with two independent variables, it is always possible to find an integration factor $\eta(q_1, q_2)$, such that the ratio $d\Gamma = dz/\eta(q_1, q_2)$ is an exact differential $\oint d\Gamma = 0$, and thus there exists a surface of the form $\Gamma = \Gamma(q_1, q_2)$. Hence, in terms of this new variable Γ which makes the constriction holonomic, the Lagrangian of the particle adopts the following form:

$$\mathcal{L} = \eta(q_1, q_2)\dot{\Gamma}^2 + h_1^2\dot{q}_1^2 + h_2^2\dot{q}_2^2 - \lambda^*(\Gamma - \Gamma(q_1, q_2)). \tag{18}$$

Note that in terms of the new variable Γ, $\eta(q_1, q_2)$ plays the role of its scale factor. Finally, the surface can be constructed graphically, by rewriting equation (16) as

$$\vec{v} \cdot \left(\Gamma\hat{z} + \sqrt{\frac{\epsilon_a}{\epsilon_\perp}}\hat{n}_\perp\right) = 0 \tag{19}$$

where $\vec{v} = \dot{\Gamma}\hat{z} + h_1\dot{q}_1\hat{e}_1 + h_2\dot{q}_2\hat{e}_2$ is the tangent vector to the ray. This allow us to identify $\mu\hat{z} + \sqrt{\epsilon_a/\epsilon_\perp}\hat{n}_\perp$ as the normal vector to the surface.

4. Energy conservation

It is interesting to calculate the physical quantities involved in the Poynting theorem within the accuracy of geometrical optics. First, substitution for $\overleftrightarrow{\epsilon} \cdot \vec{e}$ from equation (4) and for $\vec{h}^*$ from equation (5) into the time averages of the magnetic and electric energy densities, $\langle w_e\rangle = D \cdot E^*/16\pi = \vec{e}^* \cdot \overleftrightarrow{\epsilon} \cdot \vec{e}/16\pi$ and $\langle w_m\rangle = \vec{B} \cdot \vec{H}^*/16\pi = \mu\vec{h}^* \cdot \vec{h}/16\pi$, gives

$$\langle w_e\rangle = \langle w_m\rangle = -\frac{1}{16\pi}\vec{e}^{\,*} \cdot (\nabla W \times \vec{h})$$

which is the same result as that of isotropic media and is valid for both TM and TE modes. Second, the time average of the Poynting vector $\langle\vec{S}\rangle = c\mathrm{Re}\,(\vec{E} \times \vec{H}^*)/8\pi$, for the TM modes, is given by

$$\langle\vec{S}_{\mathrm{TM}}\rangle = \frac{c}{8\pi}\mathrm{Re}\,(\vec{h}^* \times \overleftrightarrow{\epsilon}^{-1} \cdot \nabla W_{\mathrm{TM}} \times \vec{h}) \tag{20}$$

whose explicit form for those modes treated in section 2 is

$$\langle\vec{S}_{\mathrm{TM}}\rangle = I\left[\left(\frac{\epsilon_{33}^{-1}}{h_2}\frac{\partial W_{\mathrm{TM}}}{\partial q_2} - \frac{\epsilon_{32}^{-1}}{h_3}\frac{\partial W_{\mathrm{TM}}}{\partial q_3}\right)\hat{e}_2 + \left(\frac{\epsilon_{22}^{-1}}{h_3}\frac{\partial W_{\mathrm{TM}}}{\partial q_3} - \frac{\epsilon_{23}^{-1}}{h_2}\frac{\partial W_{\mathrm{TM}}}{\partial q_2}\right)\hat{e}_3\right] \tag{21}$$

where $I \equiv c\mu|\vec{h}|^2/8\pi$ denotes the intensity of the incident beam. Hence, $\langle \vec{S}_{\rm TM}\rangle$ is no longer parallel to the wavevector $\nabla W_{\rm TM}$ for anisotropic media, but it points in the direction of the energy propagation. In this way, denoting by $\vec{r}(s)$ the position vector of a point in a ray, considered as a function of the length of arc s, the ray equation can be written as

$$\frac{{\rm d}\vec{r}(s)}{{\rm d}s} = \frac{\langle \vec{S}\rangle}{S} \tag{22}$$

where $S = |\langle \vec{S}\rangle|$. Notice, from the definition of $\langle \vec{S}\rangle$, that the electric and magnetic fields are orthogonal to the ray at every point as should be expected. Now, to calculate the refraction index n of this medium we use the definition for the optical length of the curve $W(P_1) - W(P_2) = \int_{P_1}^{P_2} n\, {\rm d}s$, which allows us to write

$$n = \frac{{\rm d}W}{{\rm d}s} = \frac{{\rm d}\vec{r}}{{\rm d}s}\cdot\nabla W = \frac{\langle \vec{S}\rangle}{S}\cdot\nabla W. \tag{23}$$

Substitution of $\langle \vec{S}\rangle\cdot\nabla W_{\rm TM}$ from equation (9) into (23) yields

$$n_{\rm TM} = \frac{{\rm d}W_{\rm TM}}{{\rm d}s} = \frac{I}{S} \tag{24}$$

whose explicit form for the TM modes discussed in section 2 is given by

$$n_{\rm TM} = \frac{\mu}{\sqrt{\left(\frac{\epsilon_{22}^{-1}}{h_3}\frac{\partial W_{\rm TM}}{\partial q_3} - \frac{\epsilon_{23}^{-1}}{h_2}\frac{\partial W_{\rm TM}}{\partial q_2}\right)^2 + \left(\frac{\epsilon_{33}^{-1}}{h_2}\frac{\partial W_{\rm TM}}{\partial q_2} - \frac{\epsilon_{32}^{-1}}{h_3}\frac{\partial W_{\rm TM}}{\partial q_3}\right)^2}}. \tag{25}$$

Similarly, the time average of the Poynting vector for the TE modes is simply given by

$$\langle \vec{S}_{\rm TE}\rangle = \frac{c}{8\pi\mu}{\rm Re}\,(\vec{e}\times\nabla W_{\rm TE}\times\vec{e}^{\,*}) \tag{26}$$

so that $n_{\rm TE}$ can be calculated by inserting equation (26) into equation (23) and using the relation (8). It leads to

$$n_{\rm TE} = \frac{{\rm d}W_{\rm TE}}{{\rm d}s} = \frac{c|\vec{e}|^2\epsilon_{ee}}{8\pi S}. \tag{27}$$

Since from equations (26) and (8) we have that $S_{\rm TE} = c|\vec{e}|^2\sqrt{\epsilon_{ee}/\mu}/8\pi$, then $n_{\rm TE} = \sqrt{\mu\epsilon_{ee}}$ has the usual form for isotropic media. It should be noted that the fact that $\vec{S}$ is not parallel to $\nabla W_{\rm TM}$ is the behaviour expected for anisotropic media and has been widely studied for elastic propagation [10].

Finally, for a nonconducting medium where no mechanical work is done, the Poynting theorem reduces [1] to

$$\nabla\cdot\vec{S} = I\nabla\cdot(\vec{S}/I) + (\vec{S}/I)\cdot\nabla I = 0. \tag{28}$$

For the TM modes let us introduce the operator

$$\frac{\partial}{\partial\tau} \equiv \frac{1}{I}\vec{S}_{\rm TM}\cdot\nabla \tag{29}$$

where by construction τ is a parameter which specifies the position along the ray. Hence, equation (28) can be written as

$$\frac{\partial}{\partial\tau}\ln(I) = -\nabla\cdot(\vec{S}_{\rm TM}/I) \tag{30}$$

whose solution is given by

$$I = \mathrm{e}^{-\int^{\tau} \mathrm{d}\tau\, \nabla\cdot(\overrightarrow{S}_{\mathrm{TM}}/I)}. \tag{31}$$

Using equations (29) and (9), we can express τ in terms of s, that is $\mathrm{d}\tau = I\mathrm{d}W/(\overrightarrow{S}\cdot\nabla W_{\mathrm{TM}}) = \mathrm{d}W_{\mathrm{TM}} = n_{\mathrm{TM}}\mathrm{d}s$. Thus, after taking the integration limits s_1 and s_2, equation (31) takes the form

$$\frac{I_2}{I_1} = \mathrm{e}^{-\int_{s_1}^{s_2} \mathrm{d}s\, n_{\mathrm{TM}}\nabla\cdot(\vec{S}_{\mathrm{TM}}/I)} \tag{32}$$

which is known as the law intensity. We should mention that this relation allows us to express I just in terms of W_{TM} and $\overleftrightarrow{\epsilon}^{\,-1}$ components, as can be seen from equation (21). Analogously, for the TE modes we can directly find the following expression

$$\frac{I_2}{I_1} = \mathrm{e}^{-\int_{s_1}^{s_2} \mathrm{d}s\, \nabla^2 W_{\mathrm{TE}}/n_{\mathrm{TE}}} \tag{33}$$

which is similar to the known expression for isotropic media.

5. Transport equations

In section 2 the eikonal equation was derived by using Maxwell's equation (1), (2), but it may also be derived from the wave equations for the electric and magnetic field vectors given by

$$\nabla\times(\overleftrightarrow{\epsilon}^{\,-1}\cdot\nabla\times\vec{H}) - k_0^2\mu\vec{H} = 0 \tag{34}$$

$$\nabla\times(\nabla\times\vec{E}) - k_0^2\mu\,\overleftrightarrow{\epsilon}\cdot\vec{E} = 0. \tag{35}$$

Furthermore, this alternative systematic procedure provides additionally upper-order correction terms which lead to the transport equations for the amplitude fields. Substitution of equation (3) into equations (34), (35) leads to

$$\begin{aligned} 0 &= \frac{1}{(k_0 l)^2}M_M(h_\mu, W_{\mathrm{TM}} + \frac{\mathrm{i}}{k_0 l}L_M(h_\mu, W_{\mathrm{TM}} + K_M(h_\mu, W_{\mathrm{TM}} \\ 0 &= \frac{1}{(k_0 l)^2}M_E(e_\mu, W_{\mathrm{TE}}) + \frac{\mathrm{i}}{k_0 l}L_E(e_\mu, W_{\mathrm{TE}}) + K_E(e_\mu, W_{\mathrm{TE}}) \end{aligned} \tag{36}$$

where

$$K_E(e_\mu, W_{\mathrm{TE}}) = \epsilon_{ijk}\epsilon_{kn\mu}e_\mu\frac{\partial W_{\mathrm{TE}}}{\partial q_j}\frac{\partial W_{\mathrm{TE}}}{\partial q_n} - \epsilon_{i\mu}e_\mu \tag{37}$$

$$L_E(e_\mu, W_{\mathrm{TE}}) = \epsilon_{ijk}\epsilon_{kn\mu}\left[\frac{\partial W_{\mathrm{TE}}}{\partial q_j}\frac{\partial e_\mu}{\partial q_n} + \frac{\partial W_{\mathrm{TE}}}{\partial q_n}\frac{\partial e_\mu}{\partial q_j} + e_\mu\frac{\partial^2 W_{\mathrm{TE}}}{\partial q_j\partial q_n}\right] \tag{38}$$

$$M_E(e_\mu, W_{\mathrm{TM}}) = \epsilon_{ijk}\epsilon_{kn\mu}\frac{\partial^2 e_\mu}{\partial q_j\partial q_n} \tag{39}$$

$$K_M(h_\mu, W_{\mathrm{TM}}) = \epsilon_{ijk}\epsilon_{km}^{-1}\epsilon_{mn\mu}h_\mu\frac{\partial W_{\mathrm{TM}}}{\partial q_j}\frac{\partial W_{\mathrm{TM}}}{\partial q_n} - h_\mu \tag{40}$$

$$L_M(h_\mu, W_{\mathrm{TM}}) = \epsilon_{ijk}\epsilon_{km}^{-1}\epsilon_{mn\mu}\left[\frac{\partial W_{\mathrm{TM}}}{\partial q_j}\frac{\partial h_\mu}{\partial q_n} + \frac{\partial W_{\mathrm{TM}}}{\partial q_n}\frac{\partial h_\mu}{\partial q_j} + h_\mu\frac{\partial^2 W_{\mathrm{TM}}}{\partial q_j\partial q_n}\right] \tag{41}$$

and

$$M_M(h_\mu, W) = \epsilon_{ijk}\epsilon_{km}^{-1}\epsilon_{mn\mu}\frac{\partial^2 h_\mu}{\partial q_j\partial q_n}. \tag{42}$$

Here ϵ_{ijk} is the Levi-Cevita tensor. To first-order approximation we take the dominant terms of equations (34) and (35), that is $K_E(e_\mu, W) = K_M(h_\mu, W) = 0$, and arrive consistently at the eikonal equations given by equations (8) and (9). To second-order approximation the terms L_E and L_M need to be retained, and the amplitude vectors are related with the eikonals by the expressions $L_E(e_\mu, W) = 0$ and $L_M(h_\mu, W) = 0$, that is

$$0 = \epsilon_{ijk}\epsilon^{-1}_{km}\epsilon_{mn\mu}\left[\frac{\partial W_{\mathrm{TM}}}{\partial q_j}\frac{\partial h_\mu}{\partial q_n} + \frac{\partial W_{\mathrm{TM}}}{\partial q_n}\frac{\partial h_\mu}{\partial q_j} + h_\mu\frac{\partial^2 W_{\mathrm{TM}}}{\partial q_j \partial q_n}\right] \tag{43}$$

$$0 = \epsilon_{ijk}\epsilon_{kn\mu}\left[\frac{\partial W_{\mathrm{TE}}}{\partial q_j}\frac{\partial e_\mu}{\partial q_n} + \frac{\partial W_{\mathrm{TE}}}{\partial q_n}\frac{\partial e_\mu}{\partial q_j} + e_\mu\frac{\partial^2 W_{\mathrm{TE}}}{\partial q_j \partial q_n}\right]. \tag{44}$$

If we multiply equations (43) and (44) by h_i and e_i respectively, and write the resulting expressions for the TM and TE discussed in section 2, it leads to:

$$0 = \frac{\partial|\vec{h}|^2}{\partial\tau} + \left(\frac{\epsilon^{-1}_{33}}{h_2^2}\frac{\partial^2 W_{\mathrm{TM}}}{\partial q_2^2} - 2\frac{\epsilon^{-1}_{23}}{h_2 h_3}\frac{\partial^2 W_{\mathrm{TM}}}{\partial q_2 \partial q_3} + \frac{\epsilon^{-1}_{22}}{h_3^2}\frac{\partial^2 W_{\mathrm{TM}}}{\partial q_3^2}\right)|\vec{h}|^2 \tag{45}$$

$$0 = \frac{\partial|\vec{e}|^2}{\partial\tau} + \left(\frac{1}{h_2^2}\frac{\partial^2 W_{\mathrm{TE}}}{\partial q_2^2} + \frac{1}{h_3^2}\frac{\partial^2 W_{\mathrm{TE}}}{\partial q_3^2}\right)|\vec{e}|^2 \tag{46}$$

where we have used the operator $\partial/\partial\tau$ introduced by equation (29). This expressions allow us to determine $|\vec{h}|^2$ and $|\vec{e}|^2$ in terms of W and the dielectric tensor components. Here, in contrast to the usual procedure for isotropic media, we do not need an equation for the unitary vectors $\hat{h}$ and $\hat{e}$ because they are already determined by the TM and TE modes selection.

6. Examples in nematics

As an illustrative application of our formalism let us consider a nematic droplet of radius R whose configuration could be either the radial or the axial ones, for which the director can be expressed in spherical coordinates as $\hat{n} = \sin\psi(r,\theta)\hat{e}_r + \cos\psi(r,\theta)\hat{e}_\theta$, where ψ is an angle measured from $\hat{e}_\theta$ and contained in the plane defined by this vector and $\hat{e}_r$; which are the unit vectors in the directions of increasing θ and r, respectively. Because $\hat{n}$ does not depend on ϕ, it is convenient to choose $\hat{e}_\phi$ as the transverse direction. Hence, since ϵ_{11} in equation (12) for TE modes is a constant the TE ray trajectories are straight lines for both configurations.

On the other hand the TM modes are described by equation (13) with $q_2 = r$ and $q_3 = \theta$, so that its corresponding Lagrangian can be expressed as equation (15), where the constriction is given by

$$\mathrm{d}z = -\sqrt{\epsilon_a/\epsilon_\parallel}(\cos\psi\,\mathrm{d}r - \sin\psi\, r\,\mathrm{d}\theta). \tag{47}$$

We first consider the radial structure for which $\psi = \pi/2$ and we can chose for simplicity $\eta = 1$, hence the constraining surface is given by one branch of the multivalued function $\Gamma = \sqrt{\epsilon_a/\epsilon_\parallel}r\theta$, which has been plotted in figure 1 as a function of $z \equiv r\cos\theta$ and $\rho \equiv r\sin\theta$.

Because of its curvature, this constriction makes parallel rays diverge from their initial direction in such a way that the central rays are deflected more than the rest of the rays. Furthermore, besides the normal incident ray which propagates without being deviated, no one ray is able to reach a solid angle located behind the droplet; that is to say, the defocusing effect of the droplet causes the presence of a dark zone behind of the sphere (see figure 2).

Another example is the bipolar configuration for which the nematic's director (optical axis) is induced to be aligned parallel to the droplet boundary, that is to say $\hat{n}(r = R) = \hat{e}_r$. This configuration presents two topological defects (poles) on the boundary, located at $\theta = 0$

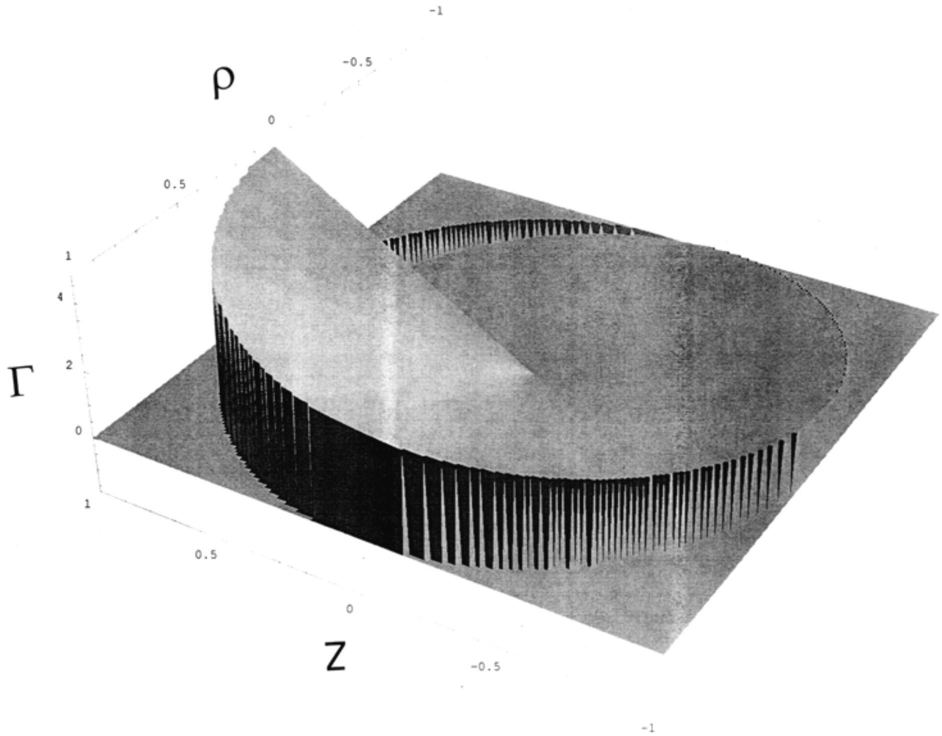

Figure 1. Constriction surface Γ versus ρ and z associated to the ray trajectories in a nematic droplet in the radial configration.

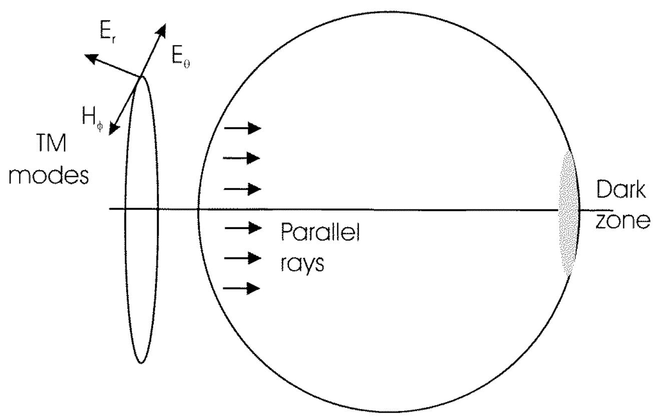

Figure 2. Schematic representation of the TM modes and a set of parallel rays in a nematic droplet. Here, the dark zone is also shown which is an unreachable region for these rays.

and $\theta = \pi$, where $\hat{n}$ is not well defined. The nematic's orientation in the whole sphere is obtained by minimizing an elastic free-energy density whose solution can be asymptotically approximated by the expression [12] $\tan\psi = (r/R - 1)\cot\theta$. Substitution of this relation into equation (47) leads to

$$d\Gamma/\Gamma = -\sqrt{\epsilon_a/\epsilon_\parallel}(-d\theta\ \tan\theta + dx\ (x-1)/x)$$

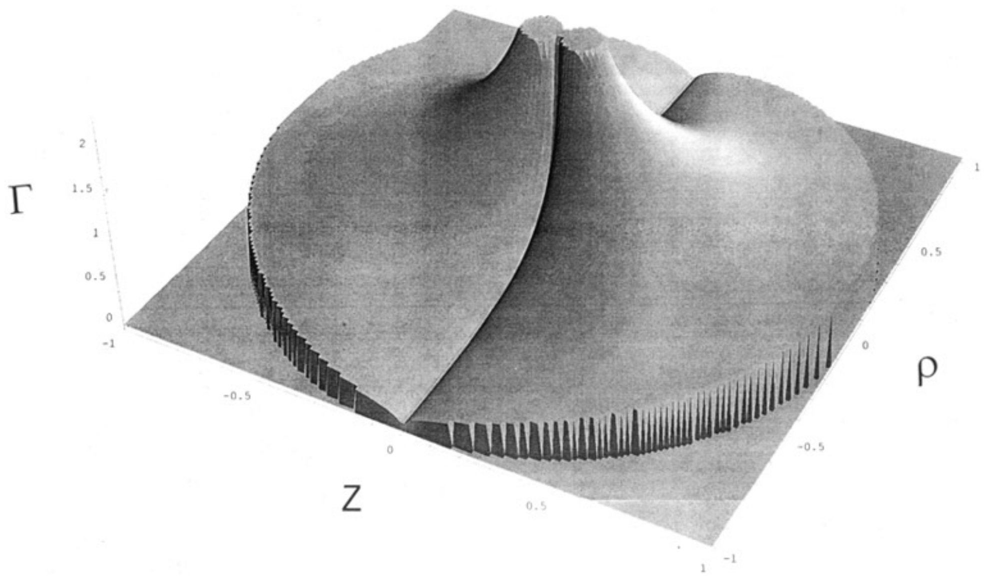

Figure 3. The same as in figure 1 but for the bipolar configuration.

where η was chosen as $\eta = (r/R\Gamma)\cos\theta/\sqrt{\sin^2\theta + (r/R-1)^2\cos^2\theta}$. Solving this differential expression we obtain the surface $\Gamma = (R\cos\theta \mathrm{e}^{-r/R}/r)^{\sqrt{\epsilon_a/\epsilon_\parallel}}$ which has the form of a screened bipolar electrostatic potential raised to the power $\sqrt{\epsilon_a/\epsilon_\parallel}$. In figure 3 we plot Γ as a function of z and ρ. In this case, the resulting trajectories are more complicated but some qualitative characteristics can be inferred. First, the rays are forced to surround the central peaks of the surface and second they are deflected when crossing the line located at $z = 0$.

We have derived the eikonal and transport equations for the TM and TE rays for a general nonmagnetic linear medium. We calculated the electromagnetic energy density, the Poynting vector, the effective index refraction and the law intensity for both the TM and the TE modes. Some analogies for interpreting their corresponding Lagrangians in terms of isotropic media were discussed. It seems that these results are relatively simple and applicable to a large number of systems.

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